MATH 1131Q: Calculus 1
Exam 1 Review Session

09/18/2024

1 Practice Problems

1. Evaluate the following limits or show they do not exist.

   (a)
   $$\lim_{x\to 0+} \frac{x^2+2x+5}{x-1}$$

   (b)
   $$\lim_{x\to 0-} \frac{x^2+2x+5}{x-1}$$

   (c)
   $$\lim_{x\to 0} \frac{x^2+2x+5}{x-1}$$

   (d)
   $$\lim_{x\to 4} \frac{x^3-16x}{x^2-x-12}$$

   (e)
   $$\lim_{x\to 3-} \frac{x^3-3x^2}{x^2-6x+9}$$

2. Find the vertical and horizontal asymptotes of the function
   $$y = \frac{x^3+1}{4x^2-12x+9}$$

3. Graph the function
   $$f(x) = \begin{cases} -2x+1 & x<-2 \\ 3 & x=-2 \\ x^2+1 & x>-2 \end{cases},$$
   and evaluate the limits $\lim_{x\to -2-} f(x)$, $\lim_{x\to -2+} f(x)$, and $\lim_{x\to -2} f(x)$. Is the function $f(x)$ continuous at -2?

4. Assume f, g are continuous functions and that $g(3) = 2$ and $\lim_{x\to 3}(4f(x) - 16f(x)g(x) + 3g(x)) = -22$. Find $f(3)$.

5. Apply the Intermediate Value Theorem to show that $f(x) = x^3 - 3x + 3$ has a root on the interval $(-4, 3)$.

6. Find the following limits or show they do not exist:

(a)
$$\lim_{x\to\infty} \frac{4x^3+1}{8x^3+2x^2+5x+6}$$

(b)
$$\lim_{x\to\infty} \frac{5x^6}{7x^4+3x}$$

(c)
$$\lim_{x\to-\infty} \frac{5x^6}{7x^4+3x}$$

(d)
$$\lim_{x\to-\infty} \frac{x^2}{16x^4-18x^2}$$

7. Let
$$f(x) = \frac{\sqrt{5x^8+1}}{x^3+2}.$$

(a) Compute
$$\lim_{x\to\infty} f(x).$$

(b) Compute
$$\lim_{x\to-\infty} f(x).$$

8. Evaluate the limit
$$\lim_{x\to 5} \frac{x-5}{\sqrt{x^2-9}-4}.$$

2 Important Concepts to Know

2.1 Slope of a secant line

- Given a function $f(x)$ and two points $x = a, x = b$, the secant line to $f(x)$ is the line that passes between the two points a, b.
- The slope of the secant line to $f(x)$ is given by
$$\frac{f(b)-f(a)}{b-a}.$$

2.2 Slope of the tangent line

Given a function $f(x)$ and a point $x = a$, the slope of the tangent line to $f(x)$ is given by either one of the expressions below:
$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}, \text{ or } \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

2.3 Limits

- The limit of a function $f(x)$ exists at a point a if the one-sided limits are equivalent at that point, i.e.,

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$$

  If the one-sided limits at the point $x = a$ disagree, then the limit *does not exist*.

- We say the function $f(x)$ is *continuous* at a point $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

- We say the function $f(x)$ is *continuous from the left at a* if

$$\lim_{x\to a^-} f(x) = f(a).$$

- We say the function $f(x)$ is *continuous from the right at a* if

$$\lim_{x\to a^+} f(x) = f(a).$$

2.3.1 Procedure for determining whether/when a function is continuous (in general)

1. Determine any one-sided limits of $f(x)$, if appropriate/necessary.
2. If you don't know what the limit of $f(x)$ is at the point due to some unknown variables, set the one-sided limits equal to each other and solve for those unknowns.
3. Once you determine any unknowns, then determine the value of $f(x)$ at the point in question (i.e., if your point is α, check $f(\alpha)$.)
4. Once you determine $f(\alpha)$, check whether $\lim_{x\to\alpha} f(x) = f(\alpha)$.
   (a) If $\lim_{x\to\alpha} f(x) = f(\alpha)$, then $f(x)$ is **continuous** at α.
   (b) If $\lim_{x\to\alpha} f(x) \neq f(\alpha)$, then $f(x)$ is **not continuous** at α.

2.3.2 Determining the behavior of a limit at $\pm\infty$

1. For a rational function, determine what the highest order term is in the numerator and in the denominator, respectively.
2. Factor out the highest order term in the numerator from the numberator, and also factor out the highest order term of the denominator from the denominator.
3. Simplify any terms and proceed, trying to evaluate the limit.

Remember that

$$\lim_{x\to\infty} \frac{1}{x^n} = 0, \text{ for any } n > 0.$$

2.4 Continuous Functions and Discontinuities

- Some examples of functions that are continuous everywhere: constant functions, polynomials, exponential functions
- Some examples of functions that are continuous wherever they are defined: trigonometric functions, logarithms
- There are several different types of discontinuities:
  1. Jump discontinuities
  2. Removable discontinuities
  3. Vertical asymptotes

2.4.1 Asymptotes

- The vertical line $x = a$ is called a vertical asymptote of the function $y = f(x)$ if at least one of the following statements is true:
  1. $\lim_{x\to a} f(x) = \infty$
  2. $\lim_{x\to a-} f(x) = \infty$
  3. $\lim_{x\to a+} f(x) = \infty$
  4. $\lim_{x\to a} f(x) = -\infty$
  5. $\lim_{x\to a-} f(x) = -\infty$
  6. $\lim_{x\to a+} f(x) = -\infty$
- A function $f(x)$ has a vertical asymptote at $y = c$ if either

$$\lim_{x\to\infty} f(x) = c, \text{ or } \lim_{x\to-\infty} f(x) = c$$

- To determine whether a rational function has a vertical asymptote, check where the denominator approaches 0.

2.5 Intermediate Value Theorem

Theorem 2.1 (Intermediate Value Theorem)**.** *Suppose that f is continuous on $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a number c in (a, b) such that $f(c) = N$.*

1. Evaluate the following limits or show they do not exist.

(a)
$$\lim_{x\to 0+}\frac{x^2+2x+5}{x-1}$$

(b)
$$\lim_{x\to 0-}\frac{x^2+2x+5}{x-1}$$

(c)
$$\lim_{x\to 0}\frac{x^2+2x+5}{x-1}$$

(d)
$$\lim_{x\to 4}\frac{x^3-16x}{x^2-x-12}$$

(e)
$$\lim_{x\to 3-}\frac{x^3-3x^2}{x^2-6x+9}$$

(1a) $\displaystyle\lim_{x\to 0^+}\frac{x^2+2x+5}{x-1} = \frac{0^2+2(0)+5}{0-1} = \frac{5}{-1} = -5$

(1b) $\displaystyle\lim_{x\to 0^-}\frac{x^2+2x+5}{x-1} = \frac{0^2+2(0)+5}{0-1} = \frac{5}{-1} = -5$

Because $\displaystyle\lim_{x\to 0^+}\frac{x^2+2x+5}{x-1} = -5 = \lim_{x\to 0^-}\frac{x^2+2x+5}{x-1}$

Then the limit as $x\to 0$ of $\frac{x^2+2x+5}{x-1}$ exists and equals to -5.

The limit of a function at a point $x=a$ exists if both of the one-sided limits at that point agree.

(1c) $\displaystyle\lim_{x\to 0}\frac{x^2+2x+5}{x-1} = \frac{0^2+2(0)+5}{0-1} = \frac{5}{-1} = -5$

(1d) $\displaystyle\lim_{x\to 4}\frac{x^3-16x}{x^2-x-12}$

First evaluate it to check if it gives you $\frac{0}{0}$.

$$\frac{4^3-16(4)}{4^2-4-12} = \frac{64-64}{16-4-12} = \frac{0}{0}.$$ This is bad.

Now, we try to simplify and reduce

$$\lim_{x\to 4}\frac{x^3-16x}{x^2-x-12} = \lim_{x\to 4}\frac{x(x^2-16)}{(x-4)(x+3)}$$

$$= \lim_{x\to 4}\frac{x\cancel{(x-4)}(x+4)}{\cancel{(x-4)}(x+3)}$$

$$= \lim_{x\to 4}\frac{x(x+4)}{x+3}$$

$$= \frac{4(4+4)}{4+3}$$

$$= \frac{4(8)}{7} = \frac{32}{7}$$

$x^2-x-12=0$ $\to$ quadratic formula

$\parallel$

$(x-4)(x+3)$

$$x = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-12)}}{2}$$

$$\Rightarrow x = \frac{1\pm\sqrt{1+48}}{2} = \frac{1\pm\sqrt{49}}{2}$$

$$= \frac{1\pm 7}{2}$$

$\frac{1+7}{2} = \frac{8}{2} = 4$ $\quad$ $\frac{1-7}{2} = \frac{-6}{2} = -3$

$\Rightarrow (x-4)(x-(-3)) = (x-4)(x+3)$

(e)

$$\lim_{x\to 3-} \frac{x^3 - 3x^2}{x^2 - 6x + 9}$$

$$\lim_{x\to 3^-} \frac{x^3-3x^2}{x^2-6x+9} = \lim_{x\to 3^-} \frac{x^2(x-3)}{(x-3)^2}$$

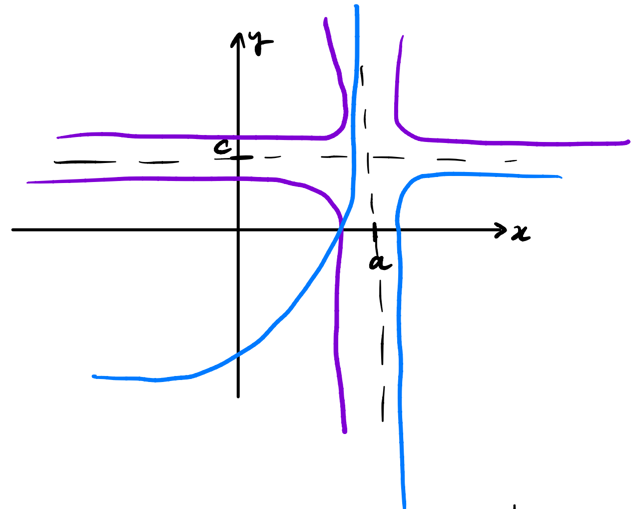
$$= \lim_{x\to 3^-} \frac{x^2}{x-3}$$

$$= -\infty$$

This limit equals $-\infty$ because as $x\to 3^-$ from the left $x-3<0$, so the whole limit approaches $-\infty$.

2. Find the vertical and horizontal asymptotes of the function

$$y = \frac{x^3+1}{4x^2-12x+9}$$



<u>vertical asymptote</u>. $x=a$ is a vertical asymptote if

$$\lim_{x\to a} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a} f(x) = -\infty$$

<u>horizontal asymptote</u>. $y=c$ is a horizontal asymptote if

$$\lim_{x\to +\infty} f(x) = c \quad \text{or} \quad \lim_{x\to -\infty} f(x) = c.$$

- When you have a rational function, one start to check for any vertical asymptotes, is to check when the denominator equals 0.

Check when $4x^2-12x+9=0$. $\xrightarrow[\text{formula}]{\text{By Quadratic}}$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2-4(4)(9)}}{2(4)}$$

$$= \frac{12 \pm \sqrt{144-16(9)}}{8}$$

$$= \frac{12 \pm \sqrt{144-144}}{8}$$

$$= \frac{12 \pm \sqrt{0}}{8}$$

$$= \frac{12}{8}$$

Since $x = \frac{12}{8}$ is a zero/root of $4x^2-12x+9$, then $\displaystyle\lim_{x\to \frac{12}{8}} \frac{x^3+1}{4x^2-12x+9} = +\infty$

In fact $x = \frac{12}{8}$ is a vertical asymptote.

<u>Horizontal asymptote</u>

$$\lim_{x\to +\infty} \frac{x^3+1}{4x^2-12x+9} = \lim_{x\to +\infty} \frac{x^3\left(1+\frac{1}{x^3}\right)}{x^2\left(4-\frac{12}{x}+\frac{9}{x^2}\right)}$$

$$= \lim_{x\to +\infty} \frac{x\left(1+\frac{1}{x^3}\right)}{4-\frac{12}{x}+\frac{9}{x^2}}$$

$$= \left(\lim_{x\to +\infty} x\right)\left(\lim_{x\to +\infty} \frac{1+\frac{1}{x^3}}{4-\frac{12}{x}+\frac{9}{x^2}}\right)$$

$$= (+\infty)\left(\frac{1}{4}\right)$$

$$= +\infty$$

$$\lim_{x\to -\infty} \frac{x^3+1}{4x^2-12x+9} = \lim_{x\to -\infty} \frac{x\left(1+\frac{1}{x^3}\right)}{4-\frac{12}{x}+\frac{9}{x^2}}$$

$$= -\infty$$

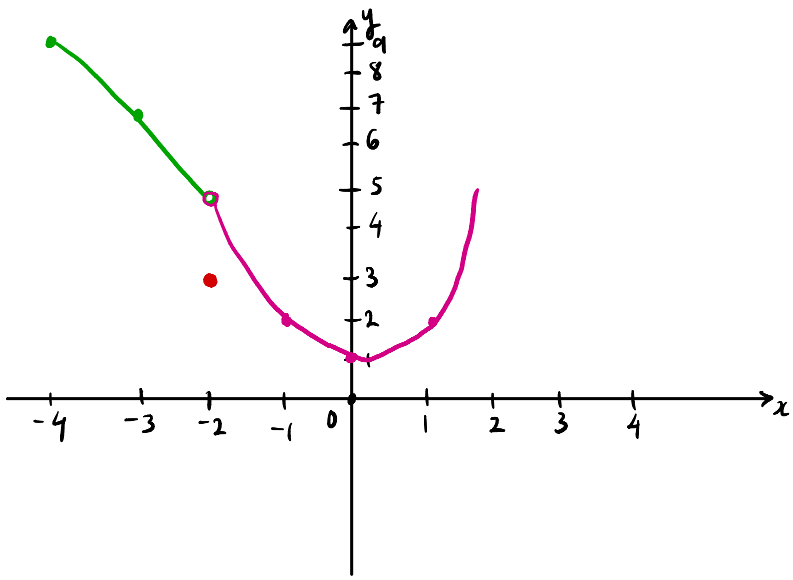
3. Graph the function

$$f(x) = \begin{cases} -2x+1 & x < -2 \\ 3 & x = -2 \\ x^2+1 & x > -2 \end{cases},$$

and evaluate the limits $\lim_{x\to -2-} f(x)$, $\lim_{x\to -2+} f(x)$, and $\lim_{x\to -2} f(x)$. Is the function $f(x)$ continuous at -2?

x	$f(x) = -2x+1$
-2	$(-2)(-2)+1 = 4+1 = 5$
-3	$(-2)(-3)+1 = 6+1 = 7$
-4	$(-2)(-4)+1 = 8+1 = 9$

x	$f(x) = x^2+1$
-2	$(-2)^2+1 = 4+1 = 5$
-1	$(-1)^2+1 = 2$
0	$0^2+1 = 1$
1	$1^2+1 = 2$



$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (-2x+1)$$

$$= (-2)(-2)+1$$

$$= 4+1$$

$$= 5$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (x^2+1)$$

$$= (-2)^2+1$$

$$= 4+1$$

$$= 5$$

Since $\lim_{x\to -2^-} f(x) = -5 = \lim_{x\to -2^+} f(x)$,

the limit $\lim_{x\to -2} f(x) = -5$ and it exists.

<u>Def</u>. A function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

We know $\lim_{x\to -2} f(x) = -5$. But, $f(-2) = 3$. Since $3 \neq -5$, $\lim_{x\to -2} f(x) \neq f(2)$,

f is <u>not</u> continuous at $x = -2$.

4. Assume f, g are continuous functions and that $g(3) = 2$ and $\lim_{x\to 3}(4f(x) - 16f(x)g(x) + 3g(x)) = -22$. Find $f(3)$.

Since f, g are continuous, $4f(x) - 16f(x)g(x) + 3g(x)$ is also continuous, and

$\lim_{x\to 3} f(x) = f(3)$ and $\lim_{x\to 3} g(x) = g(3)$.

$$\lim_{x\to 3}\left[4f(x) - 16f(x)g(x) + 3g(x)\right] = 4\lim_{x\to 3} f(x) - 16\lim_{x\to 3} f(x)g(x) + 3\lim_{x\to 3} g(x)$$

This is where we use that f, g are continuous

$$= 4f(3) - 16f(3)g(3) + 3g(3)$$

use $g(3) = 2$

$$= 4f(3) - 16f(3)(2) + 3(2)$$

$$= 4f(3) - 32f(3) + 6$$

$$= -22$$

Main Idea

use the definition of continuity.

$$\Rightarrow \quad 4f(3) - 32f(3) + 6 = -22$$

$$\Rightarrow \quad -28f(3) + 6 = -22$$

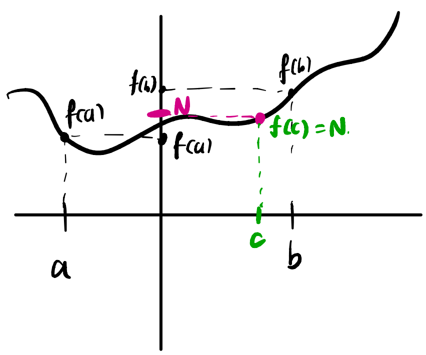
$$\Rightarrow \quad -28f(3) = -22 - 6 = -28$$

$$\Rightarrow \quad -28f(3) = -28$$

$$\Rightarrow \quad f(3) = 1.$$

5. Apply the Intermediate Value Theorem to show that $f(x) = x^3 - 3x + 3$ has a root on the interval $(-4, 3)$.

Intermediate Value Theorem: If f is continuous on $[a,b]$, and N is any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, Then there exists a number c in (a,b) such that $f(c) = N$.



To show that $f(x) = x^3 - 3x + 3$ has a root, this means there exists an x_0 in $(-4,3)$ such that $f(x_0) = x_0^3 - 3x_0 + 3 = 0$.

N.

① Is f continuous on $[-4,3]$?

Yes, because f is a polynomials and polynomials are continuous everywhere

② What is $f(-4)$ and $f(3)$?

$$f(-4) = (-4)^3 - (3)(-4) + 3 = -64 + 12 + 3 = -64 + 15 = -49$$

$$f(3) = 3^3 - 3(3) + 3 = 27 - 9 + 3 = 21$$

③ Is $N=0$ between $f(-4)$ and $f(3)$?

$\left.\begin{array}{l} f(-4) = -49 \\ f(3) = 21 \end{array}\right\}$ Is 0 between -49 and 21? Yes. $-49 < 0 < 21$.

So, by the Intermediate Value theorem, there exists a constant c in $(-4,3)$ such that $f(c) = 0$.

So, f has a root on the interval $(-4,3)$.

6. Find the following limits or show they do not exist:

(a)

$$\lim_{x\to\infty} \frac{4x^3+1}{8x^3+2x^2+5x+6}$$

(b)

$$\lim_{x\to\infty} \frac{5x^6}{7x^4+3x}$$

(c)

$$\lim_{x\to-\infty} \frac{5x^6}{7x^4+3x}$$

(d)

$$\lim_{x\to-\infty} \frac{x^2}{16x^4-18x^2}$$

(6b) $\lim\limits_{x\to+\infty} \dfrac{5x^6}{7x^4+3x} = \lim\limits_{x\to+\infty} \dfrac{5x^6}{x^4\left(7+\frac{3}{x^3}\right)}$

$= \lim\limits_{x\to+\infty} \dfrac{5x^2}{7+\frac{3}{x^3}}$

$= \left(\lim\limits_{x\to+\infty} \dfrac{5}{7+\frac{3}{x^3}}\right)\left(\lim\limits_{x\to+\infty} x^2\right)$

$= \dfrac{5}{7} \lim\limits_{x\to+\infty} x^2$

$= \dfrac{5}{7}(+\infty)$

$= +\infty$

$\lim\limits_{x\to+\infty} 7+\dfrac{3}{x^3} = 7+\lim\limits_{x\to+\infty}\dfrac{3}{x^3}$

$= 7+0$

$= 7$

(6c) $\lim\limits_{x\to-\infty} \dfrac{5x^6}{7x^4+3x} = \lim\limits_{x\to-\infty} \dfrac{5x^2}{7+\frac{3}{x^3}}$

$= \left(\lim\limits_{x\to-\infty} x^2\right)\left(\lim\limits_{x\to-\infty} \dfrac{5}{7+\frac{3}{x^3}}\right)$

$= \dfrac{5}{7}(+\infty)$

$= +\infty$

$\lim\limits_{x\to-\infty} 7+\dfrac{3}{x^3} = 7+\lim\limits_{x\to-\infty}\dfrac{3}{x^3}$

$= 7-0$

$= 7$

(a)

$$\lim_{x\to\infty} \frac{4x^3+1}{8x^3+2x^2+5x+6}$$

First, factor out the highest order power from the numerator and the denominator, respectively.

$$\lim_{x\to\infty} \frac{4x^3+1}{8x^3+2x^2+5x+6} = \lim_{x\to\infty} \frac{x^3\left(4+\frac{1}{x^3}\right)}{x^3\left(8+\frac{2}{x}+\frac{5}{x^2}+\frac{6}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{4+\frac{1}{x^3}}{8+\frac{2}{x}+\frac{5}{x^2}+\frac{6}{x^3}}$$

But $\lim_{x\to\infty}\left(4+\frac{1}{x^3}\right) = 4+\lim_{x\to\infty}\frac{1}{x^3} = 4+0 = 4$, and $\lim_{x\to\infty}\left(8+\frac{2}{x}+\frac{5}{x^2}+\frac{6}{x^3}\right) = 8+\lim_{x\to\infty}\left(\frac{2}{x}\right)+\lim_{x\to\infty}\left(\frac{5}{x^2}\right)+\lim_{x\to\infty}\left(\frac{6}{x^3}\right)$,

$$= 8+0+0+0$$

$$= 8$$

So,

$$\lim_{x\to\infty} \frac{4x^3+1}{8x^3+2x^2+5x+6} = \lim_{x\to\infty} \frac{4+\frac{1}{x^3}}{8+\frac{2}{x}+\frac{5}{x^2}+\frac{6}{x^3}}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

7. Let

$$f(x) = \frac{\sqrt{5x^8+1}}{x^3+2}.$$

(a) Compute

$$\lim_{x\to\infty} f(x).$$

(b) Compute

$$\lim_{x\to-\infty} f(x).$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$\sqrt{x^8} = (x^8)^{1/2} = x^{\frac{8}{2}} = x^4$$

(a)
$$\lim_{x\to\infty} \frac{\sqrt{5x^8+1}}{x^3+2} = \lim_{x\to\infty} \frac{\sqrt{x^8\left(5+\frac{1}{x^8}\right)}}{x^3\left(1+\frac{2}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^8}\sqrt{5+\frac{1}{x^8}}}{x^3\left(1+\frac{2}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{x^4\sqrt{5+\frac{1}{x^8}}}{x^3\left(1+\frac{2}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{x\sqrt{5+\frac{1}{x^8}}}{1+\frac{2}{x^3}}$$

$$= \left(\lim_{x\to\infty} x\right)\left(\lim_{x\to\infty} \frac{\sqrt{5+\frac{1}{x^8}}}{1+\frac{2}{x^3}}\right)$$

$$= (+\infty)\left(\frac{\sqrt{5}}{1}\right)$$

$$= +\infty$$

8. Evaluate the limit
$$\lim_{x\to 5} \frac{x-5}{\sqrt{x^2-9}-4}.$$

First multiply the numerator and the denominator by the conjugate of $\sqrt{x^2-9}-4$, which is $-\sqrt{x^2-9}-4$.

So,

$$\lim_{x\to 5} \frac{x-5}{\sqrt{x^2-9}-4} = \lim_{x\to 5} \frac{x-5}{\sqrt{x^2-9}-4} \cdot \frac{-\sqrt{x^2-9}-4}{-\sqrt{x^2-9}-4}$$ ← multiply by the conjugate

$$= \lim_{x\to 5} \frac{(x-5)(-\sqrt{x^2-9}-4)}{(-4)^2-(\sqrt{x^2-9})^2}$$ ← in the denominator, used the difference of squares formula, i.e., $(a-b)(a+b)=a^2-b^2$.

$$= \lim_{x\to 5} \frac{(x-5)(-\sqrt{x^2-9}-4)}{16-(x^2-9)}$$

$$= \lim_{x\to 5} \frac{(x-5)(-(\sqrt{x^2-9}+4))}{16-x^2+9}$$ ← factored out -1 from the term $-\sqrt{x^2-9}-4$

$$= \lim_{x\to 5} \frac{(5-x)(\sqrt{x^2-9}+4)}{25-x^2}$$ ← Multiplied $(-1)(x-5)=5-x$

$$= \lim_{x\to 5} \frac{\cancel{(5-x)}(\sqrt{x^2-9}+4)}{\cancel{(5-x)}(5+x)}$$ ← divided the $5-x$ term because it was a common factor in the numerator and the denominator.

$$= \lim_{x\to 5} \frac{\sqrt{x^2-9}+4}{5+x}$$

$$= \frac{\sqrt{5^2-9}+4}{5+5}$$

$$= \frac{\sqrt{25-9}+4}{10}$$

$$= \frac{\sqrt{16}+4}{10}$$

$$= \frac{4+4}{10}$$

$$= \frac{8}{10}$$

$$= \frac{4}{5}$$

Evaluated and simplified the limit.